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ELASTIC AND THERMOELASTIC CHARACTERISTICS OF COMPOSITES REINFORCED WITH UNIDIRECTIONAL FIBRE LAYERS*

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Composites reinforced with infinite, unidirectional cylindrical fibres are considered. It is assumed that the thermoelastic characteristics of every fibre are functions of the distance from its axis of symmetry. The material of the medium and the fibres is transversally iostropic, with the axis of isotropy coinciding with the direction of the reinforcing fibres. The problem of thermoelasticity for a medium containing an isolated, cylindrically inhomogeneous inclusion in constant external stress and temperature fields is considered first. The problem is solved using the method developed in /1/. An efficient numerical algorithm for constructing the solution is given for the case of a cylindrically layered fibre.

The interaction between the fibres in the composite is described using the effective (selfconsistent) field method /2, 3/. Tensors of effective moduli of elasticity and coefficients of linear expansion of the composities reinforced with cylindrically layered fibres are constructed. Formulas for estimating the concentrations of microstresses in the fibres within the composite are obtained, and some numerical results are given.

1. Isolated cylindrical inclusion in a homogeneous elastic medium. Let a homogeneous elastic medium with tensor of the moduli c_0 contain an inclusion occupying the region V, which has the form of an infinite circular cylinder. We shall assume that the tensor of the moduli of elasticity of the inclusion $c = c_0 + c_1$ is a piecewise-smooth function of the coordinate r, i.e. of the distance from its axis of symmetry. We shall consider the position of the point x of the medium in the Cartesian system of coordinates (x_1, x_2, x_3) and the cylindrical system of coordinates (r, n, z). Here the x_3 and z axes are directed along the axis of symmetry of the inclusion, and n is a unit vector of the r axis. If the external stress field σ_0 applied to the medium is constant, then the deformation tensor $\varepsilon(x)$ in the medium with an inclusion does not depend on the coordinate $x_3(z)$ and can be written in the form

$$\varepsilon(y) = [I + A(y)]\cdot\varepsilon_0,\ \varepsilon_0 = c_0^{-1}\cdot\sigma_0,\ y = y(x_1, x_2) \tag{1.1}$$

where a dot denotes the contraction of the tensors over two indices and I is a tetravalent unit tensor. The tensor $A(y)$, vanishing as $|y| \to \infty$, satisfies an integral equation which follows from the equation for the deformation tensor in a medium containing an inhomogeneity /4/, and has the form

$$A(y) + \int_V K(y-y')\cdot c_1(y)'\cdot A(y)'\,dy' = -\int_V K(y-y')\cdot c_1(y')\,dy' \tag{1.2}$$

The kernel $K(y)$ of the integral operator $\mathbf{K}$ in this equation is given in terms of the second derivative of Green's function $G(x)$ for a homogeneous three-dimensional medium c_0

$$K_{\alpha\beta\lambda\mu}(y) = -\int_{-\infty}^{\infty} \nabla_{\alpha)}\nabla_{(\beta}G_{\lambda)(\mu}(y, x_3)\,dx_3$$

From this it follows that the Fourier transformation $K^*(\xi)$ of the function $K(y)$ (the symbol of the operator $\mathbf{K}$) has the form

$$K^*(\xi) = K^*(k)|_{k_3=0},\quad K^*_{\alpha\beta\lambda\mu}(k) = k_{\alpha)}k_{(\beta}G^*_{\lambda)(\mu}(k)$$
$$G^*(k) = L^{-1}(k),\quad L^{\alpha\beta}(k) = k_\lambda c_0^{\lambda\alpha\beta\mu}k_\mu,\quad \xi = \xi(k_1, k_2) \tag{1.3}$$

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Here $k(k_1, k_2, k_3)$ is the vector parameter of the Fourier transformation, and the system of coordinates k_1, k_2, k_3 is conjugate to x_1, x_2, x_3. From (1.3) we see that $K^*(\xi)$ is a homogeneous function of zero degree in ξ. A solution of Eq.(1.2) exists and is unique, provided that $\det[c_0 + c_1(y)]$ is not equal to zero or infinity for all values of y /4, 5/.

2. ***Special tensor bases.*** The tensor functions appearing in the problem can be represented more easily by introducing three tensor bases. We shall use the unit vectors e and n (unit vectors of the z and r axes) as the generators of these bases, as well as a bivalent tensor $\theta_{\alpha\beta} = \delta_{\alpha\beta} - e_\alpha e_\beta$ which is a projector on the θ plane orthogonal to e ($\delta_{\alpha\beta}$ is the Kronecker delta).

First we shall determine the P-basis consisting of the following six tensors:

$$P_{1\alpha\beta\lambda\mu} = \theta_{\lambda)(\alpha}\theta_{\beta)(\mu}, \quad P_{2\alpha\beta\lambda\mu} = \theta_{\alpha\beta}\theta_{\lambda\mu}, \quad P_{3\alpha\beta\lambda\mu} = \theta_{\alpha\beta}e_\lambda e_\mu$$
$$P_{4\alpha\beta\lambda\mu} = e_\alpha e_\beta\theta_{\lambda\mu}, \quad P_{5\alpha\beta\lambda\mu} = \theta_{\lambda)(\alpha}e_{\beta)}e_\mu, \quad P_{6\alpha\beta\lambda\mu} = e_\alpha e_\beta e_\lambda e_\mu \tag{2.1}$$

The tensors P_i form a system closed under the operation of multiplication, i.e. contraction over two indices (/6/, example 4). The basis P_i is convenient for representing the tensor of the moduli of elasticity of a transversely isotropic body. In particular, the tensor c can be written in the form

$$c = \lambda P_2 + 2\mu P_1 + \tau(P_3 + P_4) + \delta P_5 + \rho P_6 \tag{2.2}$$
$$\lambda = \frac{1}{2}\left(\frac{1}{E_3\Delta} - \frac{E_1}{1+\nu_{12}}\right), \quad \mu = \frac{E_1}{2(1+\nu_{12})}, \quad \tau = \frac{\nu_{13}}{E_3\Delta}$$
$$\delta = 4\mu_3 \quad \rho = \frac{1-\nu_{12}}{E_1\Delta}, \quad \Delta = \frac{1-\nu_{12}}{E_1E_3} - 2\left(\frac{\nu_{13}}{E_3}\right)^2$$

where $\lambda, \mu, \tau, \delta, \rho$ are the scalar elasticity parameters depending on the coordinate r, and the relations connecting them with the "technical" moduli are represented by the above equations.

Here E_1 is Young's modulus in a plane transverse to the cross-section of the fibre, E_3 is the same modulus in the direction of the axis of the fibre, ν_{12}, ν_{13} are Poisson's ratios, and μ_3 is the shear modulus. A representation analogous to (2.2) (with parameters $\lambda_0, \mu_0, \tau_0, \delta_0, \rho_0$) admits of the tensor of the moduli of elasticity of the medium c_0. When the medium is isotropic, λ and μ are the Lamé parameters and $\tau = \lambda$, $\delta = 4\mu$, $\rho = \lambda + \mu$.

We form the Θ-basis from six tensors Θ_i belonging to the θ plane

$$\Theta_{1\alpha\beta\lambda\mu} = \theta_{\lambda)(\alpha}\theta_{\beta)(\mu}, \quad \Theta_{2\alpha\beta\lambda\mu} = \theta_{\alpha\beta}\theta_{\lambda\mu}, \quad \Theta_{3\alpha\beta\lambda\mu} = \theta_{\alpha\beta}n_\lambda n_\mu$$
$$\Theta_{4\alpha\beta\lambda\mu} = n_\alpha n_\beta\theta_{\lambda\mu}, \quad \Theta_{5\alpha\beta\lambda\mu} = \theta_{\lambda)(\alpha}n_{\beta)}n_{(\mu}, \quad \Theta_{6\alpha\beta\lambda\mu} = n_\alpha n_\beta n_\lambda n_\mu \tag{2.3}$$

The tensors also form a closed algebra relative to the operation of multiplication introduced above. We note that the linear spaces stretched over the P- and Θ-bases have a non-empty intersection, since $\Theta_1 = P_1$, $\Theta_2 = P_2$.

We construct the R-basis for the following five tensors:

$$R_{1\alpha\beta\lambda\mu} = n_\alpha n_\beta n_\lambda n_\mu, \quad R_{2\alpha\beta\lambda\mu} = e_\alpha e_\beta e_\lambda e_\mu$$
$$R_{3\alpha\beta\lambda\mu} = e_\alpha e_\beta n_\lambda n_\mu, \quad R_{4\alpha\beta\lambda\mu} = n_\alpha n_\beta e_\lambda e_\mu, \quad R_{5\alpha\beta\lambda\mu} = n_{(\alpha}e_{\beta)}n_{(\lambda}e_{\mu)} \tag{2.4}$$

and note that the product of the elements of the P-, Θ- and R-bases is a linear combination of the tensors (2.1), (2.3) and (2.4).

Using the relations (1.3) and (2.2), we can show that the symbol $K^*(\xi)$ of the operator **K** in (1.2) can be written in the form

$$K^*(m) = \mu_0^{-1}[\Theta_5(m) - \varkappa_0\Theta_6(m)] + 4\delta_0^{-1}R_5(m, e)$$
$$m = \xi/|\xi|, \quad \varkappa_0 = (\lambda_0 + \mu_0)/(\lambda_0 + 2\mu_0) \tag{2.5}$$

3. ***The solution of Eq.(1.2).*** In order to construct the solution of Eq.(1.2) we shall use a special representation of the integral operator **K**. The Mellin transformation of the tensor function $f(n, r)$ with respect to the variable r, has the form

$$f^*(n, s) = \int_0^\infty f(n, r)r^{s-1}dr, \quad f(n, r) = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} f^*(n, s)r^{-s}ds$$

It can be shown that the operator $\mathbf{K}$, on the piecewise smooth functions $f(n, r)$, has the representation /7/

$$(\mathbf{K}f)(n, r) = \frac{1}{2\pi i}\int_{\tau - i\infty}^{\tau + i\infty} r^{-s}(\mathbf{K}_s f^*)(n, s)\, ds \tag{3.1}$$

$$(\mathbf{K}_s f^*)(n, s) = -\frac{1}{4\pi^2}\Gamma(2 - s)\Gamma(s)\int_{\Omega_1}(-n\cdot m)^{-s} dm \int_{\Omega_1}(l\cdot m)^{s-2} K^*(m)\cdot f^*(l, s)\, dl \tag{3.2}$$

Here $\Gamma(s)$ is a gamma function and n, m, l are vectors on the unit circle Ω_1. The tensor $K^*(m)$ has the form (2.5).

Acting with the operator of Mellin transformation on both sides of the initial Eq.(1.2) and taking into account (3.1), we obtain

$$A^*(n, s) + \mathbf{K}_s(c_1\cdot A)^*(n, s) = -(\mathbf{K}_1 c_1^*)(n, s) \tag{3.3}$$

We shall seek the Mellin transform $A^*(n, s)$ of the tensor $A(n, r)$ in the form of an expansion over the elements of the P-, Θ- and R-bases with scalar coefficients depending on the parameter s. First we inspect the result of the action of the operator $\mathbf{K}_s$ on these bases. Carrying out the integration in (3.2), we can show that the following relations hold:

$$\mathbf{K}_s P_1 = \frac{1}{\mu_0(2 - s)(4 - s)}[T_1^* + (1 - \varkappa_0)T_3^*], \quad \mathbf{K}_s P_2 = \frac{1 - \varkappa_0}{\mu_0(2 - s)} T_2^* \tag{3.4}$$

$$\mathbf{K}_s P_3 = \frac{1 - \varkappa_0}{\mu_0(2 - s)} T_4^*, \quad \mathbf{K}_s P_4 = \mathbf{K}_s P_6 = 0, \quad \mathbf{K}_s P_5 = \frac{2}{\delta_0(2 - s)} T_5^*$$

The five tensors T_i^* appearing in them have the form

$$T_1^* = (4 - s)(\Theta_1 - s\Theta_5) - T_3^*, \; T_2^* = \Theta_2 - s\Theta_4 \tag{3.5}$$

$$T_3^* = \Theta_2 + 2\Theta_1 - s(\Theta_3 + \Theta_4 + 4\Theta_5) + s(2 + s)\Theta_6$$

$$T_4^* = P_3 - sR_4, \; T_5^* = P_5 - sR_5$$

The action of the operator $\mathbf{K}_s$ on the elements of the Θ-basis is expressed by three tensors given here, namely T_1^*, T_2^* and T_3^*, according to the formulas

$$\mathbf{K}_s\Theta_i = K_s P_i, \quad i = 1, 2, \quad \mathbf{K}_s\Theta_3 = \frac{1 - \varkappa_0}{\mu_0 s(2 - s)(4 - s)}[(4 - s)T_2^* - (2 - s)T_3^*] \tag{3.6}$$

$$\mathbf{K}_s\Theta_4 = \frac{(1 - \varkappa_0)(s - 1)}{\mu_0 s(2 - s)} T_2^*, \quad \mathbf{K}_s\Theta_5 = \frac{1}{2\mu_0 s(2 - s)(4 - s)}[2(1 - \varkappa_0)(s - 1)T_3^* +$$

$$sT_1^*], \quad \mathbf{K}_s\Theta_6 = \frac{s - 1}{\mu_0 s(4 - s^2)(4 - s)}\{(1 - \varkappa_0)[(4 - s)T_2^* + sT_3^*] + 2T_1^*\}$$

Finally, the action of $\mathbf{K}_s$ on the elements of the R-basis is determined by the relations

$$\mathbf{K}_s R_1 = \mathbf{K}_s\Theta_6, \quad \mathbf{K}_s R_2 = \mathbf{K}_s R_3 = 0 \tag{3.7}$$

$$\mathbf{K}_s R_4 = \frac{(1 - \varkappa_0)(s - 1)}{\mu_0 s(2 - s)} T_4^*, \quad \mathbf{K}_s R_5 = \frac{2(s - 1)}{\delta_0 s(2 - s)} T_5^*$$

Let us now turn our attention to relation (3.3). Since the tensor c_1^* on the right-hand side has a structure analogous to (2.2) with the coefficients ($\lambda_1^*(s)$, $\mu_1^*(s)$, $\tau_1^*(s)$, $\delta_1^*(s)$, $\rho_1^*(s)$, it follows from (3.4) that the right-hand side of (3.3) is a linear combination of tensors T_i^* of the form (3.5). If we now take the tensor $A(n, r)$ as a linear combination of the elements of the P-, Θ- and R-bases with coefficients depending on r, then the product $c_1 A$ will be represented in the form of the analogous expansion and the tensor $\mathbf{K}_s(c_1\cdot A)^*$ will be a linear combination of the tensors T_i^* Therefore it is natural to seek the tensor $A^*(n, s)$ in the form of a linear combination of not all the elements of the P-, Θ- and R-bases, but of must five tensors $T_i^*(n, s)$

$$A^*(n, s) = \sum_{i=1}^{5} \alpha_i^*(s) T_i^*(n, s) \tag{3.8}$$

Here $\alpha_i^*(s)$ are scalar functions of the parameter s whose r-representations are $\alpha_i(r)$ Since the operator $D = rd/dr$ in r-space corresponds to the multiplying factor $(-s)$ in the space of Mellin transformations, it follows from (3.5) and (3.8), that

$$A(n,r)=\sum_{i=1}^{5} T_i(n,D)\alpha_i(r) \tag{3.9}$$

where the differential operators $T_i(n,D)$ are determined by the right-hand sides of relations (3.5), provided that the parameter $(-s)$ in them is replaced by the operator D.

In order to construct the five scalar functions $\alpha_i(r)$, we substitute expressions (3.8) and (3.9) for $A^*(n,s)$ and $A(n,r)$ into (3.3), and use formulas (3.4)-(3.7). Here both sides of Eq.(3.3) are found to be linear combinations of five tensors $T_i^*(n,s)$. Comparing the coefficients accompanying like tensors on the left-hand and right-hand sides of the resulting equation, we arrive at the system of relations connecting the function $\alpha_i^*(s)$ After algebraic reduction, these relations take the form

$$\begin{gathered}\mu_0 s(4-s^2)(4-s)\alpha_1^*(s)+S_1^*(s)=-2s(2+s)\mu_1^*(s)\\ \mu_0 s(4-s^2)(4-s)[\alpha_3^*(s)-(1-\varkappa_0)\alpha_1^*(s)]+S_2^*(s)=0\\ (\lambda_0+2\mu_0)s(2-s)\beta^*(s)+S_3^*(s)=-2s[\lambda_1^*(s)+\mu_1^*(s)]\\ (\lambda_0+2\mu_0)s(2-s)\alpha_4^*(s)+S_4^*(s)=-s\tau_1^*(s)\\ \delta_0 s(2-s)\alpha_5^*(s)+S_5^*(s)=-2s\,\delta_1^*(s),\ \beta=\alpha_2+(4+D)\alpha_3\end{gathered} \tag{3.10}$$

Here $S_i^*(s)$ are scalar functions whose s- and r-representations have the form

$$\begin{gathered}S_1=D(2-D)\mu_1[(2+D)^2\alpha_1+4(1+D)\alpha_3]-4(1+\\ D)\mu_1 D(D-2)(\alpha_3-\alpha_1)\\ S_2=(2+D)\{(2-D)[\lambda_1 D(4+D)\alpha_3+\mu_1(6D\alpha_3+\\ D^2\alpha_1-2D\alpha_1)]-2(1+D)\mu_1 D(D-2)(\alpha_3-\alpha_1)\}\\ S_3=D[\lambda_1(2+D)+2\mu_1]\beta-2(1+D)\mu_1 D\beta\\ S_4=D\tau_1(2+D)\alpha_4-2(1+D)\mu_1 D\alpha_4\\ S_5={}^1/_2[D\delta_1-(D+1)\delta_1 D]\alpha_5\end{gathered} \tag{3.11}$$

Replacing, the functions $\alpha_i^*(s)$ and $S_i^*(s)$ in relations (3.10) by their originals (3.11) and the factor $(-s)$ by the operator D, we arrive at a system of the ordinary fourth-order differential equations for determining the functions α_1 and α_3, and of second-order differential equations for the functions β, α_4 and α_5.

If the elasticity parameters of the inclusion are picewise-smooth functions of r with derivatives equal to zero at $r=0$, then the boundary conditions for determining the functions $\alpha_i(r)$ and $\beta(r)$ will take the form

$$\begin{gathered}D\alpha_i=D^2\alpha_i=0,\ i=1,3;\ D\alpha_4=D\alpha_5=D\beta=0\\ \text{when } r=0\\ \alpha_i,\ \beta\to 0 \text{ as } r\to\infty,\ i=1,3,4,5\end{gathered} \tag{3.12}$$

The first group of these conditions holds by virtue of the continuity of the functions $A(n,r)$ and $r=0$, and the second group by virtue of the fact that $A(n,r)$ tends to zero at infinity.

4. The problem of thermoelasticity. We shall assume that the medium and the inclusion have, in addition to the moduli of elasticity, different coefficients of linear expansion α_0 and α. For transversely isotropic materials α_0 and α are two-valued tensors of the form ($\otimes$ is a tensor product)

$$\alpha_0=\alpha_{0\theta}\theta+\alpha_{0e}e\otimes e,\quad \alpha(r)=\alpha_\theta(r)\theta+\alpha_e(r)e\otimes e$$

where $\alpha_{0\theta}$ and α_θ are coefficients of linear expansion in the phase orthogonal to the axis of the fibre, and α_{0e}, α_e are the same quantities in the direction of the fibre axis. Let us consider the state of stress of the medium with an inhomogeneous cylindrical inclusion, in a constant temperature field t, assuming that the medium is free from internal stresses at $t=0$.

Let $\sigma_t(y)$ be the temperature stress tensor, $\varepsilon_t^e=c^{-1}\cdot\sigma_t$ the elastic deformation corresponding to σ_t, $\varepsilon_t=\varepsilon_t^e+\alpha t$ the total deformation ($c=c_0$, $\alpha=\alpha_0$ for $y\notin V$), $\varepsilon_{t1}=\varepsilon_t-\alpha_0 t$ is the perturbation in the total deformation related to the presence of an inhomogeneity, and $\varepsilon_{t1}(y)\to 0$ as $|y|\to\infty$.

It can be shown that the tensor $\varepsilon_{t1}(y)$ in a medium with an infinite cylindrical inclusion satisfies the following integral equation /8/:

$$\begin{gathered}\varepsilon_{t1}(y)+\int_V K(y-y')\cdot c_1(y')\cdot\varepsilon_{t1}(y')\,dy'=\\ \int_V K(y-y')\cdot c(y')\,\alpha_1(y')t\,dy'\quad(\alpha_1=\alpha-\alpha_0)\end{gathered} \tag{4.1}$$

where the kernel $K(y)$ has the same form as in (1.2). In what follows, we shall assume that $t = 1$, therefore the expressions which will be obtained for the deformations and stresses will have to be multiplied by the actual value of the temperature t.

Acting with the operator of Mellin transformation on both sides of (4.1), we obtain

$$\varepsilon_{t1}^*(n, s) + \mathbf{K}_s(c_1 \cdot \varepsilon_{t1})^*(n, s) = \mathbf{K}_s(c\ \alpha_1)^*(n, s) \tag{4.2}$$
$$c_1 \cdot \alpha_1 = (2k\alpha_{1\theta} + \tau\alpha_{1e})\,\theta + (2\tau\alpha_{1\theta} + \rho\alpha_{1e})\ e \otimes e\ (k = \lambda + \mu)$$

The operator $\mathbf{K}_s$ is given by (3.2), and its action on the tensors θ, $e \otimes e$ and $n \otimes n$ is described by the relations

$$\mathbf{K}_s\theta = \frac{1 - \varkappa_0}{\mu(2 - s)} H(n, s), \quad \mathbf{K}_s(e \otimes e) = 0 \tag{4.3}$$
$$\mathbf{K}_s(n \otimes n) = \frac{(1 - \varkappa_0)(s - 1)}{\mu_0 s(2 - s)} H(n, s), \quad H(n, s) = \theta - sn \otimes n$$

The above relations show that the functions $\varepsilon_{t1}^*(n, s)$ and $\varepsilon_{t1}(n, r)$ can be sought in the form

$$\varepsilon_{t1}^*(n, s) = (\theta - sn \otimes n)\beta_t^*(s), \quad \varepsilon_{t1}(n, r) = (\theta + n \otimes nD)\,\beta_t(r) \tag{4.4}$$

where $\beta_t^*(s)$ is the Mellin transform of the scalar function $\beta_t(r)$. Substituting (4.4) into (4.2) we can confirm that the left- and right-hand sides of (4.2) are proportional to the tensor $H(n, s)$, and comparing the coefficients of this tensor on the left- and right-hand sides we obtain

$$k_0 s(2 - s)\,\beta_t^*(s) + S_t^*(s) = \varkappa_0 s\gamma^*(s) \tag{4.5}$$
$$\gamma(r) = 2k(r)\,\alpha_{1\theta}(r) + \tau(r)\,\alpha_{1e}(r), \quad k_0 = \lambda_0 + \mu_0$$
$$S_t(r) = \varkappa_0\,[D\,(2k_1(r) + \lambda_1(r)\,D) - 2\,(D + 1)\,\mu_1(r)\,D]\beta_t(r)$$

Replacing the functions $S_t^*(s)$, $\beta_t^*(s)$, $\gamma^*(s)$ in (4.5) by their originals and the factor $(-s)$ by the operator D, we obtain an ordinary second-order differential equation for $\beta_t(r)$. The boundary conditions for this equation will have a form analogous to (3.12)

$$D\beta_t \quad 0 \text{ when } r = 0;\ \beta_t \to 0 \text{ as } r \to \infty$$

5. *Cylindrically layered inclusion.* Let the elasticity parameters and the coefficients of linear expansion of the inclusion be piecewise-constant functions with discontinuities at the points $r = a_i$, $i = 1, 2, \ldots, N$, $0 < a_1 < a_2 < \ldots < a_N$. In this case the inclusion will consist of a kernel and $N - 1$ cylindrical layers. In the regions within which the properties are constant (within the layers) the differential equations for the functions α_j, β and β_t, which follow from (3.10) and (4.5), simplify and take the form

$$D\,(2 - D)(2 + D)(4 + D)\,\alpha_j = 0, \quad j = 1,3; \quad D\,(2 + D)\,\alpha_j = 0,$$
$$j = 4,5$$
$$D\,(2 + D)\,\beta = 0, \quad D\,(2 + D)\,\beta_t = 0$$

This implies that the form of the functions α_j, β and β_t is described, within the range $a_{i-1} < r < a_i$, $i = 1, 2, \ldots, N$ $(a_0 = 0,\ a_{N+1} = \infty)$, by the relations

$$\alpha_1 = Y_1{}^i + Y_2{}^i r^2 + Y_3{}^i r^{-2} + Y_4{}^i r^{-4}, \quad \alpha_3 = Y_5{}^i + Y_6{}^i r^2 + Y_7{}^i r^{-2} + Y_8{}^i r^{-4} \tag{5.1}$$
$$\beta = Y_9{}^i + Y_{10}{}^i r^{-2}, \quad \alpha_4 = Y_{11}{}^i + Y_{12}{}^i r^{-2}, \quad \alpha_5 = Y_{13}{}^i + Y_{14}{}^i r^{-2},$$
$$\beta_t = Y_{15}{}^i + Y_{16}{}^i r^{-2}$$

where $Y_j{}^i$ are arbitrary constants. Thus the solution of the problem within each layer is determined, apart from 16 constants.

Let us inspect the jumps in the derivatives of the functions $\alpha_j(r)$, $\beta(r)$ and $\beta_t(r)$ at the boundaries of the layers when $r = a_i$. We denote by $[\varphi]_i$ the jump in the piecewise-smooth function $\varphi(r)$ at the point $r = a_i$

$$[\varphi]_i = \varphi(a_i + 0) - \varphi(a_i - 0), \quad \varphi(a_i \pm 0) = \lim_{\varepsilon \to 0} \varphi(a_i \pm \varepsilon), \quad \varepsilon > 0$$

Using relations (3.10) and (4.5) in the same manner as in /1/, we can establish the validity of the following relations for the functions $\alpha_j(r)$ $\beta(r)$ and $\beta_t(r)$ of the form (5.1)

$$[\alpha_j]_i = 0,\ j = 1, 3, 4, 5,\ [\beta]_i = [\beta_t]_i = 0,\ [D\alpha_j]_i = 0,\ j = 1, 3 \tag{5.2}$$
$$[\mu D^2\alpha_1]_i = -2\,[\mu]_i - 4\,[\mu\alpha_1]_i - 2\,[\mu D\alpha_1]_i - 4[\mu\alpha_3]_i - 4\,[\mu D\alpha_3]_i$$

$$[\mu D^3\alpha_1]_i = 12\,[\mu]_i + 24\,[\mu\alpha_1]_i + 16\,[\mu D\alpha_1]_i + 24\,[\mu\alpha_3]_i + 12\,[\mu D\alpha_3]_i$$
$$[(\lambda + 2\mu)D^2\alpha_3]_i = -2\,[\mu]_i - 4\,[\mu\alpha_1]_i - 4[\mu D\alpha_1]_i - 4\,[\mu\alpha_3]_i - 2\,[(2\lambda + 3\mu)\,D\alpha_3]_i$$

$$[(\lambda + 2\mu)D^3\alpha_3]_i = 12\,[\mu]_i + 24\,[\mu\alpha_1]_i + 12\,[\mu D\alpha_1]_i + 24\,[\mu\alpha_3]_i + 16\,[(\lambda + 2\mu)\,D\alpha_3]_i$$
$$[(\lambda + 2\mu)D\beta]_i = -2\,[(\lambda + \mu)(1 + \beta)]_i,\ [\delta D\alpha_5]_i = -[\delta\,(2 + \alpha_5)]_i$$
$$[(\lambda + 2\mu)D\alpha_4]_i = -[\tau]_i - 2\,[(\lambda + \mu)\alpha_4]_i,\ \ [(\lambda + 2\mu)D\beta_t]_i = [\gamma]_i - 2\,[(\lambda + \mu)\,\beta_t]_i$$
$$(i = 1, 2, \ldots, N)$$

Using relations (5.1), (5.2) and boundary conditions (3.12), (4.6), we can find the whole bulk of the constants Y_j^i determining the solution of the problem in question. Let us describe the algorithm for calculating these constants.

We introduce $N + 1$ 16-dimensional vectors Y^i whose components will be the constants appearing in (5.1), and $N + 1$ vectors $X^i(r)$ with components

$$\begin{gathered}X_1^i = \alpha_1,\ X_2^i = D\alpha_1,\ X_3^i = D^2\alpha_1,\ X_4^i = D^3\alpha_1,\ X_5^i = \alpha_3,\\ X_6^i = D\alpha_3,\ X_7^i = D^2\alpha_3,\ X_8^i = D^3\alpha_3,\ X_9^i = \beta,\ X_{10}^i = D\beta,\ X_{11}^i = \\ \alpha_4,\ X_{12}^i = D\alpha_4,\ X_{13}^i = \alpha_5,\ X_{14}^i = D\alpha_5,\ X_{15}^i = \beta_t,\ X_{16}^i = D\beta_t\\ \alpha_j = \alpha_j\,(r), \beta = \beta\,(r), \beta_t = \beta_t\,(r),\ a_{i-1} < r < a_i,\ i = 1, 2, \ldots, N + 1\end{gathered} \tag{5.3}$$

The vectors Y^i and X^i are connected by the relations which follow from (5.1)

$$\begin{gathered}X^i\,(r) = H\,(r)\,Y^i,\ \ Y^i = H^{-1}\,(r)\,X^i\,(r),\\ H = \overset{2}{\oplus}\,h_1 \overset{4}{\oplus}\,h_2\\ h_1 = \begin{Vmatrix}1 & r^2 & r^{-2} & r^{-4}\\ 0 & 2r^2 & -2r^{-2} & -4r^{-4}\\ 0 & 4r^2 & 4r^{-2} & 16r^{-4}\\ 0 & 8r^2 & -8r^{-2} & -64r^{-r}\end{Vmatrix},\quad h_2 = \begin{Vmatrix}1 & r^{-2}\\ 0 & -2r^{-2}\end{Vmatrix}\end{gathered} \tag{5.4}$$

where $\overset{n}{\oplus}$ is the direct (Cartesian) n-tuple matrix product.

From (5.4) it follows that the value of the vector $X^i(r)$ at the right end of the i-th interval $(r = a_i - 0)$ can be expressed in terms of its value at the left and $(r = a_{i-1} + 0)$ by the formulas

$$X^i\,(a_i) = R^iX^i\,(a_{i-1}),\quad R^i = H\,(a_i)\,H^{-1}\,(a_{i-1}),\quad i = 2, 3, \ldots, N \tag{5.5}$$

By virtue of (5.2) the vectors X^i and X^{i+1} at the point $r = a_i$ at the boundary of the i-th and $(i + 1)$-th layer are connected by the relations

$$X^{i+1}\,(a_i + 0) = F^i + \Gamma^i X^i\,(a_i - 0) \tag{5.6}$$

where the form of the matrix Γ^i and vector F^i is reestablished from (5.2).

Let the vector of the solution in the first layer $X^1\,(a_1)$ be known. Then the vector $X^{i+1}\,(a_i)$, determining the solution in the $(i + 1)$-th layer will be found from the formulas which follow from (5.5) and (5.6)

$$\begin{gathered}X^{i+1}\,(a_i) = g^i + G^iX^1\,(a_1),\quad i = 1, 2, \ldots, N\\ g^1 = F^1,\quad g^i = F^i + \sum_{j=1}^{i}\Big(\prod_{k=i}^{j+1} Q^k\Big)F^j,\quad i \geqslant 2\\ G^i = \prod_{k=i}^{1} Q^k,\quad Q^k = \Gamma^kR^k\end{gathered} \tag{5.7}$$

Here R^1 is a unit matrix, and the matrices R^k and $k \geqslant 2$ are definded in (5.5). We use the boundary conditions of the problem to construct the vector $X^1\,(a_1)$. From the fact that the solution is bounded at $r = 0$ and tends to zero at infinity, it follows that expressions (5.1) for the functions α_j, β and β_t in the first interval $(0 < r < a_1)$ contain no negative powers of r, and no positive powers of r in the $(N + 1)$-th interval $(a_N < r < \infty)$, i.e. $Y_j^1 = 0$ for $j = 3, 4, 7, 8, 10, 12, 14, 16$; $Y_j^{N+1} = 0$ for $j = 1, 2, 5, 6, 9, 11, 13, 15$ Using the methods described in /1, 8/ we can obtain from this an equation for determining the vector $X^1\,(a_1)$

$$BZ = f,\ X^1(a_1) = MZ \tag{5.8}$$

$$B = LG^N M, \quad f = -Lg^N, \quad m_1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \\ 0 & 4 \end{Vmatrix}, \quad m_2 = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{Vmatrix}$$

$$M = \overset{2}{\oplus} m_1 \overset{2}{\oplus} m_2,$$

$$L = \overset{2}{\oplus} l_1 \overset{4}{\oplus} l_2, \quad l_1 = \begin{Vmatrix} 8 & 6 & 1 & 0 \\ -48 & -28 & 0 & 1 \end{Vmatrix}, \quad l_2 = \| 2 \quad 1 \|$$

Here B, M, L are the 8×8, 16×8 and 8×16 matrices respectively, Z is an eight-dimensional vector, and G^N and g^N are defined in (5.7).

Having determined the vector $X^1(a_1)$ from (5.8), we use (5.7) to obtain all vectors $X^{i+1}(a_i)$, $i = 1, 2, \ldots, N$, and (5.4) to obtain the constants Y_j^i which determine the solution of the problem in every interval $a_{i-1} < r < a_i$, $i = 1, 2, \ldots N + 1$.

Figs.1 and 2 show graphs of the distribution of the stress σ_{11} along the x_2 axis in a medium with a cylindrically inhomogeneous inclusion in the case when a uniaxial stress σ_0 acts along the x_1 axis. When σ_{11} was computed according to the proposed scheme, the medium and the inclusion were both assumed to be isotropic with the same Poisson's ratio $\nu = 0.4$, and Young's modulus varying according to the law

$$E(r) = E_0\left[1 + \delta \exp\left(\frac{\lambda r^2}{r^2 - 1}\right)\right] \text{ when } r < 1, \quad E(r) = E_0 \text{ when } r \geqslant 1 \tag{5.9}$$

The inclusion was partitioned in the course of computation into N layers of thickness $1/N$, and Young's modulus of the i-th layer was assumed to be equal to $E(i/N)$ and computed from (5.9). When $N > 40$ the distribution of the stresses did not depend on the number of partitions and corresponded to a continuous variation in the value of the modulus in accordance with (5.9). The graphs in Fig.1 correspond to a compliant inclusion ($\delta = -0.99$), and those in Fig.2 to a rigid inclusion ($\delta = 100$).

6. A medium containing a set of cylindrically layered fibres. Let us now consider a medium containing a set of identical, cylindrically layered inclusions. We shall assume that all inclusions have the same orientation and that their distribution within the plane of transverse cross-section of the fibres is statistically isotropic. In order to construct the thermoelastic characteristics of such a medium, we shall use the effective (selfconsistent) field method, based on the solution of the problem of an isolated fibre obtained earlier. The scheme of the method was described in /1-3, 8/, and can be used here without any alterations.

Let us give the final expressions for the tensors of the effective moduli of elasticity c_* and the coefficients of linear expansion α_* of the composite material, obtained by means of the method mentioned above:

$$c_* = c_0 + \Lambda_s \cdot (I - A_0 \cdot \Lambda_s)^{-1}, \quad \alpha_* = \alpha_0 - (I - D_0 \cdot \Lambda_s)^{-1} \cdot \Lambda_t \tag{6.1}$$

$$D_0 = c_0 \cdot A_0 \cdot c_0 - c_0, \quad A_0 = b_1 P_2 + b_2 (P_1 - {}^1/_2 P_2) + b_3 P_5$$

$$b_1 = \frac{1 - \varkappa_0}{4\mu_0}, \quad b_2 = \frac{2 - \varkappa_0}{4\mu_0}, \quad b_3 = \frac{2}{\delta_0}$$

The tetravalent Λ_s and bivalent Λ_t tensors appearing here are given in terms of integrals over the cross-section ω of an arbitrary fibre, and have the form (n_0 is the numerical concentration of the inclusions)

$$\Lambda_s = n_0 \int_\omega c_1(y)\,(I + A(y))\,dy, \quad \Lambda_t = n_0 c_0^{-1} \cdot \int_\omega [c_1(y) \cdot \varepsilon_{t1}(y) - c(y)\,\alpha_1(y)]\,dy \tag{6.2}$$

where the tensors $A(y)$ and $\varepsilon_{t1}(y)$ are given by the relations (3.9) and (4.4). Solving these integrals we obtain (the summation is carried out over i from 1 to N, Y_j^i are the constants determining the solution of the problem according to (5.1))

$$\Lambda_s = q_1 P_2 + q_2 (P_1 - {}^1/_2 P_2) + q_3 (P_3 + P_4) + q_5 P_5 + q_6 P_6 \tag{6.3}$$

$$q_1 = p \sum k_1^i (1 + Y_9^i)\,\xi_i, \quad k_1^i = k^i - k_0$$

$$q_2 = 2p \sum \mu_1^i [(1 + 2Y_1^i + 2Y_5^i)\,\xi_i + 3(Y_2^i + Y_6^i)(a_i^2 + a_{i-1}^2)\,\xi_i]$$

$$q_3 = p \sum \tau_1^i (1 + Y_9^i)\,\xi_i, \quad q_5 = {}^1/_2 p \sum \delta_1^i (2 + Y_{13}^i)\,\xi_i$$

$$q_6 = p \sum (\rho_1^i + 2\tau_1^i Y_{11}^i)\,\xi_i$$

$$\Lambda_t = q_\theta\theta + q_e e \otimes e, \quad q_\theta = p\sum t_\theta{}^i\xi_i, \quad q_e = p\sum t_e{}^i\xi_i \tag{6.4}$$

$$t_\theta{}^i = \frac{\tau_0\rho_0}{\Delta_0}\left[\left(\frac{\tau_1{}^i}{\rho_0} - \frac{k_1{}^i}{\tau_0}\right)Y_{15}{}^i + \left(\frac{k^i}{\tau_0} - \frac{\tau^i}{\rho_0}\right)\alpha_{1\theta}{}^i + \frac{1}{2}\left(\frac{\tau^i}{\tau_0} - \frac{\rho^i}{\rho_0}\right)\alpha_{1e}{}^i\right]$$

$$t_e{}^i = \frac{\tau_0 k_0}{\Delta_0}\left[2\left(\frac{k_1{}^i}{k_0} - \frac{\tau_1{}^i}{\tau_0}\right)(Y_{15}{}^i - \alpha_{1\theta}{}^i) - \left(\frac{\tau^i}{k_0} - \frac{\rho^i}{\tau_0}\right)\alpha_{1e}{}^i\right]$$

$$p = \pi n_0 a_N{}^2, \quad \xi_i = (a_i{}^2 - a_{i-1}^2)/a_N{}^2, \quad \Delta_0 = k_0\rho_0 - \tau_0{}^2$$

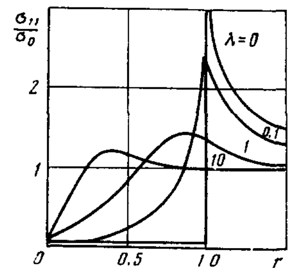
Substituting (6.3) into (6.1) we obtain

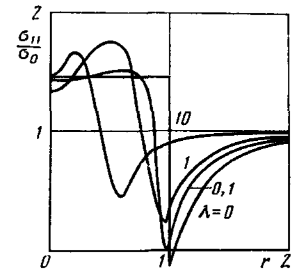
$$c_* = k_* P_2 + 2\mu_*\left(P_1 - \frac{1}{2}P_2\right) + \tau_*(P_3 + P_4) + \delta_* P_5 + \rho_* P_6 \tag{6.5}$$

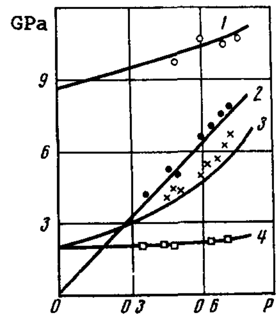
$$k_* = k_0 + \frac{q_1}{1 - 4b_1q_1}, \quad \mu_* = \mu_0 + \frac{1}{2}\frac{q_2}{1 - b_2q_2}, \quad \tau_* = \tau_0 + \frac{q_3}{1 - 4b_1q_1}$$

$$\delta_* = \delta_0 + \frac{4q_5}{4 - b_5q_5}, \quad \rho_* = \rho_0 + q_6 + \frac{4b_1q_3{}^2}{1 - 4b_1q_1}$$

We can obtain expressions for the coefficients of linear expansion $\alpha_{*\theta}$ and α_{*e} from (6.3), (6.4) and (6.1). When $N = 1$ and the matrix and the inclusions are all isotropic, the expressions for c_* and α_* become identical to those obtained in /2/.







The nature of the discrepancies arising in the averaging method used here was discussed in /2, 9/. It was noted that in the case of glass fibre-reinforced plastic-type materials the relative error in computing c_* and α_* from formulas analogous to (6.1) did not exceed 10...15% up to a concentration of the fibres close to dense packing. Fig.3 compares the values of the effective elasticity parameters of the fibrous composite calculated using (6.5), and the experimental results obtained in /10/. Composites based on an isotropic epoxide resin matrix ($E_0 = 5{,}27$ GPA, $\nu_0 = 0{,}3$) and containing transversely isotropic carbon fibres ($E_1 = 8$ GPa, $E_3 = 410\,6$ GPa, $\nu_{12} = 0.568$, $\nu_{13} = 0.277$, $\mu_3 = 10.2$ GPa) were investigated in /10/. Curves *1-4* correspond to the computed dependence of the parameters $\lambda_* + 2\mu_*$, $\rho_*/40$, $\delta_*/4$, μ_* on the concentration of the fibres p. The experimental values are shown by the light and dark dots, crosses and squares.

In conclusion we note that the effective-field method enables us to estimate the concentration of the microstresses on separate fibres within the composite, using a well-known scheme /11/. The stress tensor in the neighbourhood of a typical fibre acted upon by an external field σ_0 can be calculated using the formula

$$\sigma(y) = c(y)\cdot(I + A(y))\cdot\varepsilon_{*s}, \quad \varepsilon_{*s} = [I - \Lambda_s\cdot(A_0 - I)]^{-1}\cdot\varepsilon_0$$

and when the composite is heated (cooled) to the temperature t, we have

$$\sigma(y) = c(y)\cdot[\varepsilon_{t1}(y) - \alpha_1(y) + (I + A(y))\cdot\varepsilon_{*t}]\,t$$

$$\varepsilon_{*t} = (I - D_0\cdot\Lambda_s)^{-1}\cdot D_0\cdot c_0{}^{-1}\cdot\Lambda_t$$

Here ε_{*s} and ε_{*t} are the effective external deformation fields within which a typical inclusion is found. By introducing these fields the interaction between the fibres within the composite is taken into account.

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THE PRESSURE EXERTED BY A STAMP OF CIRCULAR CROSS-SECTION ON AN ELASTIC HALF-SPACE*

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A solution of the problem of a circular stamp in its exact formulation, i.e. without simplifying assumptions regarding satisfaction of the boundary conditions and Laplace's equation, is obtained. A method of solving three-dimensional contact problems of the theory of elasticity due to Mossakovskii is used, and the solution obtained is compared with the solution given in /1/.

As we know /2/, the problem of the pressure exerted by a stamp of circular cross-section reduces, in the case of axial symmetry, to determining the normal derivative $F_z'(\rho, 0)$ in the region of contact, and the function $F(\rho, z)$ harmonic in the half-space and vanishing at infinity, which satisfies the following boundary conditions:

$$F_z'(\rho, 0) = 0, \ 0 < \rho < a, \ b < \rho < \infty, \ F(\rho, 0) = f(\rho) \ \ a < \rho < \infty \tag{1}$$

where a and b denote the inner and outer radius of the annulus, ρ is the polar radius, and $z = f(\rho)$ is the equation of the stamp surface (the z axis is directed into the elastic half-space).

The pressure under the stamp $P(\rho)$ is given by the formula

$$P(\rho) = 1/2E(1 - \nu^2)^{-1} F_z'(\rho, 0), \ a < \rho < b$$

where E is the modulus of elasticity and ν is Poisson's ratio.

In the general case we must assume that

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